Counting zeros of holomorphic functions of exponential growth

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Abstract

We consider the number of zeros of holomorphic functions in a bounded domain that depend on a small parameter and satisfy an exponential upper bound near the boundary of the domain and similar lower bounds at finitely many points along the boundary. Roughly the number of such zeros is $(2\pi h)^{-1}$ times the integral over the domain of the laplacian of the exponent of the dominating exponential. Such results have already been obtained by M. Hager and by Hager and the author and they are of importance when studying the asymptotic distribution of eigenvalues of elliptic operators with small random perturbations. In this paper we generalize these results and arrive at geometrically natural statements and natural remainder estimates.

Résumé

Nous étudions le nombre de zéros dans un domaine borné d'une fonction holomorphe dépendant d'un petit paramètre et vérifiant une borne exponentielle supérieure près du bord et des bornes exponentielles inférieures dans un nombre fini de points près du bord. Approximativement le nombre de ces zéros est égale à $(2\pi h)^{-1}$ fois l'intégrale sur le domaine du laplacien appliqué à l'exposant de la borne exponentielle supérieure. De tels résultats ont déjà été obtenus par M. Hager et par Hager et l'auteur dans des études sur la distribution asymptotique

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des valeurs propres d'opérateurs elliptiques avec des petites perturbations aléatoires. Dans ce travail on généralise ces résultats pour arriver à des énoncés géométriquement naturels avec des estimations du reste naturelles.

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Contents

1 Introduction

Starting with the works of M. Hager [10, 8, 9] there has been a number of results ([11, 17, 18, 6, 2, 3]) that establish Weyl asymptotics of the eigenvalues of non-self-adjoint (pseudo)differential operators with small random perturbations, in the semi-classical limit and in the limit of large eigenvalues. A common feature here (as well as in many other non-self-adjoint spectral problems) is that one identifies the eigenvalues with the zeros of a holomorphic function $u(z) = u(z;h)$, $0 < h \ll 1$ in a set $\Gamma \Subset \mathbf{C}$. The available information is an upper bound $|u(z;h)| \le \exp(\phi(z)/h)$ for z near the boundary $\partial\Gamma$ as well as lower bounds $|u(z_j;h)| \ge \exp(\phi(z_j) - \epsilon_j)/h$, for finitely many points $z_j = z_j(h)$, $1 \le j \le N(h)$ that are suitably distributed near the boundary.

Hager [10, 8, 9] obtained a result of this type with the conclusion that the number of zeros in Γ is $(2\pi h)^{-1}(\int_\Lambda \Delta\phi(z)L(dz) + o(1))$ in the limit of small h, when Γ is independent of h with smooth boundary and ϕ is a C^2 function, also independent of h. In [11] we generalized this result by weakening the regularity assumptions on ϕ. However, due to some logarithmic losses, we were not quite able to recover Hager's original result, and we still had a fixed domain Γ with smooth boundary.

In many spectral problems of the above type the domain should be allowed to depend on h, for instance, it could be a long thin rectangle, and the boundary regularity should be relaxed.

In the present paper we have revisited systematically the proof of the counting proposition in [11] and obtained a general and quite natural result allowing an h -dependent exponent ϕ to be merely continuous and the h-dependent domain Γ to have Lipschitz boundary. The result generalizes the two earlier ones. By allowing suitable small changes of the points z_j we also get rid of the logarithmic losses. The new results below allow some improvement in the spectral results of [18] (see for instance [19]) and they seem to allow a better understanding of such results in general. We hope that future works will supply applications.

We next formulate the results. Let $\Gamma \Subset \mathbf{C}$ be an open set and let $\gamma = \partial\Gamma$ be the boundary of Γ. Let $r : \gamma \to]0, \infty[$ be a Lipschitz function of Lipschitz modulus $\leq 1/2$:

$$|r(x) - r(y)| \leq \frac{1}{2}|x - y|,\ \ x, y \in \gamma. \tag{1.1}$$

We further assume that γ is Lipschitz in the following precise sense, where r enters:

There exists a constant C_0 such that for every $x \in \gamma$ there exist new affine coordinates $\widetilde{y} = (\widetilde{y}_1, \widetilde{y}_2)$ of the form $\widetilde{y} = U(y - x)$, $y \in \mathbf{C} \simeq \mathbf{R}^2$ being the old coordinates, where $U = U_x$ is orthogonal, such that the intersection of Γ and the rectangle $R_x := \{y \in \mathbf{C};\ |\widetilde{y}_1| < r(x),\ |\widetilde{y}_2| < C_0 r(x)\}$ takes the form

$$\{y \in R_x;\ \widetilde{y}_2 > f_x(\widetilde{y}_1),\ \ |\widetilde{y}_1| < r(x)\}, \tag{1.2}$$

where $f_x(\widetilde{y}_1)$ is Lipschitz on $[-r(x), r(x)]$, with Lipschitz modulus $\leq C_0$.

Notice that our assumption (1.2) remains valid if we decrease r. It will be convenient to extend the function to all of $\mathbf{C}$, by putting

$$r(x) = \inf_{y \in \gamma} (r(y) + \frac{1}{2}|x - y|). \tag{1.3}$$

The extended function is also Lipschitz with modulus $\leq \frac{1}{2}$:

$$|r(x) - r(y)| \leq \frac{1}{2}|x - y|,\ \ x, y \in \mathbf{C}.$$

Notice that

$$r(x) \geq \frac{1}{2}\mathrm{dist}\,(x, \gamma), \tag{1.4}$$

and that

$$|y-x| \leq r(x) \Rightarrow \frac{r(x)}{2} \leq r(y) \leq \frac{3r(x)}{2}. \tag{1.5}$$

For simplicity, we shall also assume that Γ is simply connected. The complete version of our result is:

Theorem 1.1 *Let $\Gamma \Subset \mathbf{C}$ be simply connnected and have Lipschitz boundary γ with an associated Lipschitz weight r as in (1.1), (1.2), (1.3). Put $\widetilde{\gamma}_{\alpha r} = \cup_{x\in\gamma} D(x, \alpha r(x))$ for any constant $\alpha > 0$. Let $z_j^0 \in \gamma$, $j \in \mathbf{Z}/N\mathbf{Z}$ be distributed along the boundary in the positively oriented sense such that*

$$r(z_j^0)/4 \leq |z_{j+1}^0 - z_j^0| \leq r(z_j^0)/2.$$

(Here "4" can be replaced by any fixed constant > 2.) Then there exists a constant $C_1 > 0$ depending only on the constant C_0 in the assumption around (1.2) such that if $z_j \in D(z_j^0, r(z_j^0)/(2C_1))$ we have the following:

Let $0 < h \leq 1$ and let ϕ be a continuous subharmonic function on $\widetilde{\gamma}_r$ with a distribution extension to $\Gamma \cup \widetilde{\gamma}_r$ that will be denoted by the same symbol. Then there exists a constant $C_2 > 0$ such that if u is a holomorphic function on $\Gamma \cup \widetilde{\gamma}_r$ satisfying

$$h \ln |u| \leq \phi(z) \ \text{ on } \widetilde{\gamma}_r, \tag{1.6}$$

$$h \ln |u(z_j)| \geq \phi(z_j) - \epsilon_j, \ \text{ for } j = 1, 2, ..., N, \tag{1.7}$$

where $\epsilon_j \geq 0$, then the number of zeros of u in Γ satisfies

$$\begin{aligned} &|\#(u^{-1}(0) \cap \Gamma) - \frac{1}{2\pi h}\mu(\Gamma)| \leq \\ &\frac{C_2}{h}\left(\mu(\widetilde{\gamma}_r) + \sum_1^N \left(\epsilon_j + \int_{D(z_j, \frac{r(z_j)}{4C_1})} |\ln \frac{|w-z_j|}{r(z_j)}| \mu(dw)\right)\right). \end{aligned} \tag{1.8}$$

Here $\mu := \Delta\phi \in \mathcal{D}'(\Gamma \cup \widetilde{\gamma}_r)$ is a positive measure on $\widetilde{\gamma}_r$ so that $\mu(\Gamma)$ and $\mu(\widetilde{\gamma}_r)$ are well-defined. Moreover, the constant C_2 only depends on C_0 in (1.2).

By observing that the average of $|\ln \frac{|w-z_j|}{r(z_j)}|$ with respect to the Lebesgue measure $L(dz_j)$ over $D(z_j^0, \frac{r(z_j^0)}{2C_1})$ is $\mathcal{O}(1)$, we can get rid of the logarithmic terms in Theorem 1.1, to the price of making a suitable choice of $z_j = \widetilde{z}_j$, and we get:

Theorem 1.2 *Let Γ, $\gamma = \partial\Gamma$, r, z_j^0, C_0, C_1, ϕ be as in Theorem 1.1. Then $\exists\, \widetilde{z}_j \in D(z_j^0, \frac{r(z_j^0)}{2C_1})$ such that if h, u are as in Theorem 1.1, satisfying (1.6), and*

$$h \ln |u(\widetilde{z}_j)| \geq \phi(\widetilde{z}_j) - \epsilon_j,\ \ j = 1, 2, ..., N, \tag{1.9}$$

instead of (1.7), then

$$|\#(u^{-1}(0) \cap \Gamma) - \frac{1}{2\pi h}\mu(\Gamma)| \leq \frac{C_2}{h}(\mu(\widetilde{\gamma}_r) + \sum \epsilon_j). \tag{1.10}$$

Of course, if we already know that

$$\int_{D(z_j, \frac{r(z_j)}{4C_1})} |\ln \frac{|w - z_j|}{r(z_j)}|\mu(dw) = \mathcal{O}(1)\mu(D(z_j, \frac{r(z_j)}{4C_1})), \tag{1.11}$$

then we can keep $\widetilde{z}_j = z_j$ in (1.8) and get (1.10). This is the case, if we assume that μ is equivalent to the Lebesgue measure $L(d\omega)$ in the following sense:

$$\frac{\mu(dw)}{\mu(D(z_j, \frac{r(z_j)}{4C_2}))} \asymp \frac{L(dw)}{L(D(z_j, \frac{r(z_j)}{4C_2}))} \text{ on } D(z_j, \frac{r(z_j)}{4C_2}), \tag{1.12}$$

uniformly for $j = 1, 2, ..., N$.

Then we get,

Theorem 1.3 *Make the assumptions of Theorem 1.1 as well as (1.11) or the stronger assumption (1.12). Then from (1.6), (1.7), we conclude (1.10).*

In particular, we recover the counting proposition of M. Hager [8, 9], where Γ, ϕ are independent of h, γ of class C^∞ and $\phi \in C^\infty(\text{neigh}\,(\gamma))$ (and the replacement of "∞" by "2" is straight forward). Then $\mu \asymp L$ and if we choose $r \ll 1$ constant and assume (1.6), (1.7), we get from (1.9):

$$|\#(u^{-1}(0) \cap \Gamma) - \frac{1}{2\pi h}\mu(\Gamma)| \leq \frac{\widetilde{C}}{h}(r + \sum_1^N \epsilon_j). \tag{1.13}$$

Hager had $\epsilon_j = \epsilon$ independent of j, $r = \sqrt{\epsilon}$, $N \asymp \epsilon^{-1/2}$, so the remainder in (1.13) is $\mathcal{O}(\frac{\sqrt{\epsilon}}{h})$. The counting proposition in [11] can also be recovered.

There has been a considerable activity in the study of the zero set of random holomorphic functions where the Edelman Kostlan formula has similarities with the above results (and the earlier ones by Hager and others metnioned above) and where many further results have

been obtained. See M. Sodin [20], Sodin, B. Tsirelson [21], S. Zrebiec [22], B. Shiffman, S. Zelditch, S. Zrebiec [16]. The two last papers deal with holomorphic functions of several variables and it would be interest to see if our results also have extensions to the case of several variables. Our results are similar in spirit to classical results on zeros of entire functions, see Levin [14].

The outline of the paper is the following:

In Section 2 we consider thin neighborhoods of the boundary where the width is variable and determined by the function r. We verify that we can find such neighborhoods with smooth boundary and estimate the derivatives of the boundary defining function. Then we develop some exponentially weighted estimates for the Laplacian in such domains in the spirit of what can be done for the Schrödinger equation ([12]) and a large number of works in thin domains, see for instance [7, 4]. From that we also deduce pointwise estimates on the corresponding Green kernel.

In Section 3 we prove the main results by following the general strategy of the proof of the corresponding result in [11] and carry out the averaging argument that leads to the elimination of the logarithms.

In Section 4 we consider as a simple illustration the zeros of sums of exponentials of holomorphic functions. These results can also be obtained with more direct methods, cf [5, 1, 13].

Finally in Section 5 we establish a connection with classical results on zeros of entire functions.

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2 Thin neighborhoods of the boundary and weighted estimates

Let Γ, $\gamma = \partial\Gamma$, r be as in the introduction.

Using a locally finite covering with discs $D(x, r(x))$ and a subordinated partition of unity, it is standard to find a smooth function $\widetilde{r}(x)$ satisfying

$$\frac{1}{C} r(x) \le \widetilde{r}(x) \le r(x),\ |\nabla \widetilde{r}(x)| \le \frac{1}{2},\ \partial^\alpha \widetilde{r}(x) = \mathcal{O}(\widetilde{r}^{1-|\alpha|}), \tag{2.1}$$

where $C > 0$ is a universal constant.

From now on, we replace $r(x)$ by $\widetilde{r}(x)$ and the drop the tilde. (1.1), (1.2), (1.5) remain valid and (1.4) remains valid in the weakened form:

$$r(x) \ge \frac{1}{C} \mathrm{dist}\,(x, \gamma),$$

where $C>0$ is new constant.

Consider the signed distance to γ:

$$g(x)=\begin{cases}\mathrm{dist}\,(x,\gamma), & x\in\Gamma\\ -\mathrm{dist}\,(x,\gamma), & x\in\mathbf{C}\setminus\Gamma\end{cases}\tag{2.2}$$

Possibly after replacing r by a small constant multiple of r we deduce from (1.2) that for every $x\in\gamma$ there exists a normalized constant real vector field $\nu=\nu_x$ (namely $\partial_{\widetilde{y}_1}$, cf. (1.2)) such that

$$\nu(g)\geq\frac{1}{C}\text{ in }R_x,\quad C=(1+C_0^2)^{-1/2}.\tag{2.3}$$

In the set $\cup_{x\in\gamma}R_x$, we consider the regularized function

$$g_\epsilon(x)=\int\frac{1}{(\epsilon r(x))^2}\chi(\frac{x-y}{\epsilon r(x)})g(y)L(dy),\tag{2.4}$$

where $0\leq\chi\in C_0^\infty(D(0,1))$, $\int\chi(x)L(dx)=1$. Here $\epsilon>0$ is small and we notice that $r(x)\asymp r(y)$, $g(y)=\mathcal{O}(r(y))$, when $\chi((x-y)/(\epsilon r(x))\neq 0$. It follows that $g_\epsilon(x)=\mathcal{O}(r(x))$ and more precisely, since g is Lipschitz, that

$$g_\epsilon(x)-g(x)=\mathcal{O}(\epsilon r(x)).\tag{2.5}$$

Differentiating (2.4), we get

$$\begin{aligned}\nabla_x g_\epsilon(x)\;&=\;(\nabla_x g)_\epsilon+2\int\frac{-\nabla r(x)}{\epsilon^2 r(x)^2}\chi(\frac{x-y}{\epsilon r(x)})\frac{g(y)}{r(x)}L(dy)\\&\quad+\int\frac{1}{\epsilon^2 r(x)^2}\chi'(\frac{x-y}{\epsilon r(x)})\cdot\frac{x-y}{\epsilon r(x)}(-\nabla r(x))\frac{g(y)}{r(x)}L(dy),\end{aligned}\tag{2.6}$$

where $(\nabla_x g)_\epsilon$ is defined as in (2.4) with g replaced by $\nabla_x g$. It follows that

$$\nabla_x g_\epsilon(x)-(\nabla g)_\epsilon(x)=\mathcal{O}(1)\sup_{y\in D(x,\epsilon r(x))}\frac{|g(y)|}{r(x)}.\tag{2.7}$$

In particular, $\nabla_x g_\epsilon=\mathcal{O}(1)$ and with $\nu=\nu_x$:

$$\nu(g_\epsilon)(y)\geq\frac{1}{2C},\text{ when }y\in R_x,\text{ and }\sup_{|z-y|\leq\epsilon r(y)}|g(z)|\ll r(y).\tag{2.8}$$

Differentiating (2.6) further, we get

$$\partial^\alpha g_\epsilon(x)=\mathcal{O}_\alpha((\epsilon r(x))^{1-|\alpha|}),\ |\alpha|\geq 1.\tag{2.9}$$

Let $C>0$ be large enough but independent of ϵ. Put

$$\widehat{\gamma}_{\epsilon,C\epsilon r}=\{x\in\cup_{y\in\gamma}R_y;\ |g_\epsilon(x)|<C\epsilon r(x)\}.\tag{2.10}$$

If $C > 0$ is sufficiently large, then in the coordinates associated to (1.2), $\widehat{\gamma}_{\epsilon,C\epsilon r}$ takes the form

$$f_x^-(\widetilde{y}_1) < \widetilde{y}_2 < f_x^+(\widetilde{y}_1),\ |\widetilde{y}_1| < r(x), \tag{2.11}$$

where $f_x^\pm$ are smooth on $[-r(x), r(x)]$ and satisfy

$$\partial_{\widetilde{y}_1}^k f_x^\pm = \mathcal{O}_k((\epsilon r(x))^{1-k}),\ k \geq 1, \tag{2.12}$$

$$0 < f_x^+ - f_x,\ f_x - f_x^- \asymp C\epsilon r(x). \tag{2.13}$$

Later, we will fix $\epsilon > 0$ small enough and write $\gamma_r = \widehat{\gamma}_{\epsilon,C\epsilon r}$ and more generally, $\gamma_{\alpha r} = \widehat{\gamma}_{\epsilon,C\epsilon\alpha r}$.

We shall next establish an exponentially weighted estimate for the Dirichlet Laplacian in γ_r:

Proposition 2.1 *Let $C > 0$ be sufficiently large and $\epsilon > 0$ sufficiently small in the definition of γ_r. Then there exists a new constant $C > 0$ such that if $\phi \in C^2(\overline{\gamma}_r)$ and*

$$|\phi_x'| \leq \frac{1}{Cr}, \tag{2.14}$$

we have

$$\|e^\phi Du\| + \frac{1}{C}\|\frac{1}{r}e^\phi u\| \leq C\|re^\phi \Delta u\|,\ u \in (H_0^1 \cap H^2)(\gamma_r), \tag{2.15}$$

where $\|w\|$ denotes the L^2 norm when the function w is scalar and we write

$$(v|w) = \int \sum v_j(x)\overline{w}_j(x)L(dx),\ \|v\| = \sqrt{(v|v)},$$

for $\mathbf{C}^n$-valued functions with components in L^2. H_0^1 and H^2 are the standard Sobolev spaces.

Proof. Let $\phi \in C^2(\overline{\gamma}_r; \mathbf{R})$ and put

$$-\Delta_\phi = e^\phi \circ (-\Delta) \circ e^{-\phi} = D_x^2 - (\phi_x')^2 + i(\phi_x' \circ D_x + D_x \circ \phi_x'),$$

where we make the usual observation that the last term is formally anti-self-adjoint. Then for every $u \in (H^2 \cap H_0^1)(\gamma_r)$:

$$(-\Delta_\phi u|u) = \|D_x u\|^2 - ((\phi_x')^2 u|u). \tag{2.16}$$

We need an apriori estimate for D_x. Let $v : \overline{\gamma}_r \to \mathbf{R}^n$ be sufficiently smooth. We sometimes consider v as a vector field. Then for $u \in (H^2 \cap H_0^1)(\gamma_r)$:

$$(Du|vu) - (vu|Du) = i(\mathrm{div}\,(v)u|u).$$

Assume $\operatorname{div}(v) > 0$. Recall that if $v = \nabla w$, then $\operatorname{div}(v) = \Delta w$, so it suffices to take w strictly subharmonic. Then

$$\int \operatorname{div}(v)|u|^2 dx \leq 2\|vu\|\|Du\| \leq \|Du\|^2 + \|vu\|^2,$$

which we write

$$\int (\operatorname{div}(v) - |v|^2)|u|^2 dx \leq \|Du\|^2.$$

Using this in (2.16), we get

$$\frac{1}{2}\|Du\|^2 + \int (\frac{1}{2}(\operatorname{div}(v) - |v|^2) - (\phi'_x)^2)|u|^2 dx \leq$$
$$\|\frac{1}{k}(-\Delta_\phi)u\|\|ku\| \leq \frac{1}{2}\|\frac{1}{k}(-\Delta_\phi)u\|^2 + \frac{1}{2}\|ku\|^2,$$

where k is any positive continuous function on $\overline{\gamma}_r$. We write this as

$$\frac{1}{2}\|Du\|^2 + \int (\frac{1}{2}(\operatorname{div}(v) - |v|^2 - k^2) - (\phi'_x)^2)|u|^2 dx \leq \frac{1}{2}\|\frac{1}{k}(-\Delta_\phi)u\|^2. \tag{2.17}$$

We shall see that we can choose v so that

$$\operatorname{div}(v) \geq r^{-2},\ |v| \leq \mathcal{O}(r^{-1}). \tag{2.18}$$

After replacing v by $C^{-1}v$ for a sufficiently large constant C, we then achieve that

$$\operatorname{div}(v) - |v|^2 \asymp r^{-2}. \tag{2.19}$$

Before continuing, let us establish (2.18): Let $g = g_\epsilon$ be the function in the definition of $\gamma_r = \widehat{\gamma}_{\epsilon,C\epsilon r}$ in (2.10), so that $C^{-1} \leq |\nabla g| \leq 1$ (with the new C independent of ϵ, C in (2.10)), $\partial^\alpha g = \mathcal{O}_\epsilon(r(x)^{1-|\alpha|})$. Put

$$v = \nabla(e^{\lambda g/r}), \tag{2.20}$$

where $\lambda > 0$ will be sufficiently large. Notice that

$$\nabla(\frac{g}{r}) = \frac{\nabla g}{r} - \frac{g\nabla r}{r^2},$$

where

$$|\frac{\nabla g}{r}| \asymp \frac{1}{r}$$

uniformly with respect to ϵ and

$$|\frac{g\nabla r}{r^2}| = \mathcal{O}(1)\frac{g}{r}\frac{1}{r} = \mathcal{O}(\epsilon)\frac{1}{r},$$

in γ_r, so if we fix $\epsilon > 0$ sufficiently small, then

$$|\nabla(\frac{g}{r})| \asymp \frac{1}{r}.$$

We have

$$v = e^{\frac{\lambda g}{r}} \lambda \nabla(\frac{g}{r}),\ |v| \asymp e^{\lambda \mathcal{O}(\epsilon)} \frac{\lambda}{r},$$

so the second part of (2.18) holds for every fixed value of λ. Further,

$$\mathrm{div}\,(v) = e^{\frac{\lambda g}{r}} (\lambda^2 |\nabla(\frac{g}{r})|^2 + \lambda \Delta(\frac{g}{r})).$$

Here,

$$|\nabla(\frac{g}{r})|^2 \asymp \frac{1}{r^2},\ \Delta(\frac{g}{r}) = \mathcal{O}(\frac{1}{r^2}),$$

so if we fix λ large enough, we also get the first part of (2.18).

If we choose $k = (Cr)^{-1}$ for a sufficiently large constant C, we get from (2.19), (2.14)

$$\frac{1}{2}(\mathrm{div}\,(v) - |v|^2 - k^2) - (\phi'_x)^2 \asymp r^{-2}.$$

Thus, with a new sufficiently large constant C, we get from (2.17):

$$\|Du\|^2 + \frac{1}{C} \int_{\gamma_r} \frac{1}{r^2} |u|^2 dx \le C \|r(-\Delta_\phi)u\|^2, \tag{2.21}$$

which we can also write as

$$\|Du\| + \frac{1}{C} \|\frac{1}{r} u\| \le C \|r(-\Delta_\phi)u\|. \tag{2.22}$$

Keeping in mind (2.14), we get (2.15) by applying (2.22) to $e^\phi u$. $\square$

If $\Omega \Subset \mathbf{C}$ has smooth boundary, let G_Ω, P_Ω denote the Green and the Poisson kernels of Ω, so that the Dirichlet problem,

$$\Delta u = v,\ u_{|\partial\Omega} = f, \quad u, v \in C^\infty(\overline{\Omega}),\ f \in C^\infty(\partial\Omega),$$

has the unique solution

$$u(x) = \int_\Omega G_\Omega(x,y) v(y) L(dy) + \int_{\partial\Omega} P_\Omega(x,y) f(y) |dy|.$$

Recall that $-G_\Omega \ge 0$, $P_\Omega \ge 0$. It is also clear that

$$-G_\Omega(x,y) \le C - \frac{1}{2\pi} \ln |x-y|, \tag{2.23}$$

where $C > 0$ only depends on the diameter of Ω. Indeed, let $-G_0(x, y)$ denote the right hand side of (2.23) and choose $C > 0$ large enough so that $-G_0 \geq 0$ on $\Omega \times \Omega$. Then on the operator level,

$$G_\Omega v = G_0 v - P_\Omega(G_0 v_{|\partial\Omega}),$$

so that

$$G_\Omega(x, y) = G_0(x, y) - \int_{\partial\Omega} P_\Omega(x, z) G_0(z, y)|dz|,$$

and hence $G_\Omega \geq G_0$, $-G_\Omega \leq -G_0$.

We will also use the scaling property:

$$G_\Omega(\frac{x}{t}, \frac{y}{t}) = G_{t\Omega}(x, y),\ x, y \in t\Omega, t > 0, \tag{2.24}$$

and the fact that $-G_\Omega$ is an increasing function of Ω in the natural sense.

Proposition 2.2 *Under the same assumptions as in Proposition 2.1 there exists a (new) constant $C > 0$ such that we have*

$$-G_{\gamma_r}(x, y) \leq C - \frac{1}{2\pi} \ln \frac{|x - y|}{r(y)},\ \textit{when}\ |x - y| \leq \frac{r(y)}{C}, \tag{2.25}$$

$$-G_{\gamma_r}(x, y) \leq C \exp(-\frac{1}{C} \int_{\pi_\gamma(y)}^{\pi_\gamma(x)} \frac{1}{r(t)} |dt|),\ \textit{when}\ |x - y| \geq \frac{r(y)}{C}, \tag{2.26}$$

where it is understood that the integral is evaluated along γ from $\pi_\gamma(y) \in \gamma$ to $\pi_\gamma(x) \in \gamma$, where $\pi_\gamma(y)$, $\pi_\gamma(x)$ denote points in γ with $|x - \pi_\gamma(x)| = \operatorname{dist}(x, \gamma)$, $|y - \pi_\gamma(y)| = \operatorname{dist}(y, \gamma)$, and we choose these two points (when they are not uniquely defined) and the intermediate segment in such a way that the integral is as small as possible.

Proof. Let $y \in \gamma_r$, and put $t = r(y)$. Then we can find $\Omega \Subset \mathbf{C}$ uniformly bounded (with respect to y) whose boundary is uniformly bounded in the C^∞ sense, such that γ_r coincides with $y + t\Omega =: \Omega_y$ in $D(y, 2r(y)/C)$, $\Omega_y \subset D(y, \frac{4r(y)}{C})$ and $r \asymp r(y)$ in that disc. In view of (2.23), (2.24) we see that $-G_{\Omega_y}(x, y)$ satisfies the upper bound in (2.25). Let $\chi = \chi(\frac{x-y}{r(y)})$ be a standard cut-off equal to one on $D(y, \frac{r(y)}{C})$ with $\operatorname{supp} \chi(\frac{\cdot - y}{r(y)}) \subset D(y, \frac{2r(y)}{C})$, and write the identity:

$$G_{\gamma_r}(\cdot, y) = \chi(\frac{\cdot - y}{r(y)}) G_{\Omega_y}(\cdot, y) - G_{\gamma_r}[\Delta, \chi(\frac{\cdot - y}{r(y)})] G_{\Omega_y}(\cdot, y). \tag{2.27}$$

Using that $-G_{\Omega_y}$ satisfies (2.25), we see that the L^2-norm of $G_{\Omega_y}(\cdot, y)$ over the cut-off region (i.e. the support of the x-gradient of the cut-off) is $\mathcal{O}(r(y))$ and since G_{Ω_y} is harmonic with boundary value 0 there,

the L^2-norm of $\nabla_x G_{\Omega_y}(x,y)$ over the same region is $\mathcal{O}(1)$. It follows that

$$\|[\Delta,\chi(\frac{\cdot - y}{r(y)})]G_{\Omega_y}(\cdot,y)\| = \mathcal{O}(\frac{1}{r(y)}),$$

and hence, by applying (2.15) with $\phi = 0$ to

$$u = G_{\gamma_r}[\Delta,\chi(\frac{\cdot - y}{r(y)})]G_{\Omega_y}(\cdot,y),$$

we get

$$\frac{1}{r}G_{\gamma_r}[\Delta,\chi(\frac{\cdot - y}{r(y)})]G_{\Omega_y}(\cdot,y) = \mathcal{O}(1), \text{ in } L^2(\gamma_r).$$

Away from $\mathrm{supp}\,[\Delta,\chi(\frac{\cdot-y}{r(y)})]$ the function $G_{\gamma_r}[\Delta,\chi(\frac{\cdot-y}{r(y)})]G_{\Omega_y}(\cdot,y)$ is harmonic on γ_r with boundary value zero and we conclude that inside the region where $\chi(\frac{\cdot-y}{r(y)}) = 1$, it is $\mathcal{O}(1)$. From (2.27) we then get the estimate (2.25).

To get (2.26) we now apply the same reasoning to (2.27), now with ϕ as in (2.14), (2.15), together with standard arguments for exponentially weighted estimates, for instance as in [12]. $\square$

We will also need a lower bound on G_{γ_r} on suitable subsets of γ_r. For $\epsilon > 0$ fixed and sufficiently small, we say that $M \Subset \gamma_r$ is an elementary piece of γ_r if

- $M \subset \gamma_{(1-\frac{1}{C})r}$,
- $\frac{1}{C} \le \frac{r(x)}{r(y)} \le C$, $x,y \in M$,
- $\exists y \in M$ such that $M = y + r(y)\widetilde{M}$, where $\widetilde{M}$ belongs to a bounded set of relatively compact subsets of $\mathbf{C}$ with smooth boundary.

In the following, it will be tacitly understood that we choose our elementary pieces with some uniform control (C fixed and uniform control on the $\widetilde{M}$).

Proposition 2.3 *If M is an elementary piece in γ_r, then*

$$-G_{\gamma_r}(x,y) \asymp 1 + |\ln\frac{|x-y|}{r(y)}|,\ x,y \in M. \tag{2.28}$$

Proof. We just outline the argument. First, by using arguments from the proof of Proposition 2.2 (without any exponential weights), we see that

$$-G_{\gamma_r}(x,y) \asymp -\ln\frac{|x-y|}{r(y)},\ \text{when } x,y \in M,\ \frac{|x-y|}{r(y)} \ll 1. \tag{2.29}$$

Next, if M' is a slightly larger elementary piece of the form $y+(1+\frac{1}{C})r(y)\widetilde{M}$, then from Harnack's inequality for the positive harmonic function $-G_{\gamma_r}(\cdot,y)$ on $M'\setminus D(y,\frac{1}{2C}r(y))$, we see that $-G_{\gamma_r}(x,y)\asymp 1$ in $M\setminus D(y,\frac{1}{C}r(y))$, which together with (2.29) gives (2.28). $\square$

3 Distribution of zeros

Let ϕ be a continuous subharmonic function defined in some neighborhood of $\overline{\gamma_r}$. Let

$$\mu=\mu_\phi=\Delta\phi \tag{3.1}$$

be the corresponding locally finite positive measure.

Let u be a holomorphic function defined in a neighborhood of $\Gamma\cup\overline{\gamma_r}$. We assume that

$$h\ln|u(z)|\le\phi(z),\ \ z\in\overline{\gamma_r}. \tag{3.2}$$

Lemma 3.1 *Let $z_0\in M$, where M is an elementary piece, such that*

$$h\ln|u(z_0)|\ge\phi(z_0)-\epsilon,\ \ 0<\epsilon\ll 1. \tag{3.3}$$

Then the number of zeros of u in M is

$$\le\frac{C}{h}(\epsilon+\int_{\gamma_r}-G_{\gamma_r}(z_0,w)\mu(dw)). \tag{3.4}$$

Proof. Writing ϕ as a uniform limit of an increasing sequence of smooth functions, we may assume that $\phi\in C^\infty$. Let

$$n_u(dz)=\sum 2\pi\delta(z-z_j),$$

where z_j are the zeros of u counted with their multiplicity. We may assume that no z_j are situated on $\partial\gamma_r$. Then, since $\Delta\ln|u|=n_u$,

$$\begin{aligned}
h\ln|u(z)| &= \\
&\int_{\gamma_r}G_{\gamma_r}(z,w)hn_u(dw)+\int_{\partial\gamma_r}P_{\gamma_r}(z,w)h\ln|u(w)||dw| \\
&\le \int_{\gamma_r}G_{\gamma_r}(z,w)hn_u(dw)+\int_{\partial\gamma_r}P_{\gamma_r}(z,w)\phi(w)|dw| \\
&= \int_{\gamma_r}G_{\gamma_r}(z,w)hn_u(dw)+\phi(z)-\int_{\gamma_r}G_{\gamma_r}(z,w)\mu(dw).
\end{aligned} \tag{3.5}$$

Putting $z=z_0$ in (3.5) and using (3.3), we get

$$\int_{\gamma_r}-G_{\gamma_r}(z_0,w)hn_u(dw)\le\epsilon+\int_{\gamma_r}-G_{\gamma_r}(z_0,w)\mu(dw).$$

Now

$$-G_{\gamma_r}(z_0, w) \geq \frac{1}{C},\ w \in M,$$

and we get (3.4). □

Notice that this argument is basically the same as when using Jensen's formula to estimate the number of zeros of a holomorphic function in a disc.

Let z_j^0, z_j be as in Theorem 1.1. We may arrange so that $\widetilde{\gamma}_{r/C_1} \subset \gamma_r \subset \widetilde{\gamma}_r$. In particular, the assumptions of Theorem 1.1 imply (3.2). Now we sharpen the assumption (3.3) and assume as in Theorem 1.1,

$$h \ln |u(z_j)| \geq \phi(z_j) - \epsilon_j. \tag{3.6}$$

Let $M_j \subset \gamma_r$ be elementary pieces such that

$$z_j \in M_j,\ \mathrm{dist}\,(z_j, M_k) \geq \frac{r(z_j)}{C} \text{ when } k \neq j,\ \gamma_{\widetilde{r}} \subset \cup_j M_j,\ \widetilde{r} = (1 - \frac{1}{\widetilde{C}})r, \tag{3.7}$$

where $\widetilde{C} \gg 1$. Recall that $\gamma_r = \widehat{\gamma}_{\epsilon, C\epsilon r}$ where C, ϵ are now fixed (cf (2.10), and that $\gamma_{\alpha r} = \widehat{\gamma}_{\epsilon, \alpha C\epsilon r}$. We will also assume for a while that ϕ is smooth.

According to Lemma 3.1, we have

$$\#(u^{-1}(0) \cap M_j) \leq \frac{C_3}{h}(\epsilon_j + \int_{\gamma_r} -G_{\gamma_r}(z_j, w)\mu(dw)). \tag{3.8}$$

Consider the harmonic functions on $\gamma_{\widetilde{r}}$,

$$\Psi(z) = h(\ln |u(z)| + \int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z, w) n_u(dw)), \tag{3.9}$$

$$\Phi(z) = \phi(z) + \int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z, w)\mu(dw). \tag{3.10}$$

Then $\Phi(z) \geq \phi(z)$ with equality on $\partial\gamma_{\widetilde{r}}$. Similarly, $\Psi(z) \geq h \ln |u(z)|$ with equality on $\partial\gamma_{\widetilde{r}}$.

Consider the harmonic function

$$H(z) = \Phi(z) - \Psi(z),\ z \in \gamma_{\widetilde{r}}. \tag{3.11}$$

Then on $\partial\gamma_{\widetilde{r}}$, we have by (3.2) that

$$H(z) = \phi(z) - h \ln |u(z)| \geq 0,$$

so by the maximum principle,

$$H(z) \geq 0,\ \text{on } \gamma_{\widetilde{r}}. \tag{3.12}$$

By (3.6), we have

$$\begin{aligned} H(z_j) &= \Phi(z_j) - \Psi(z_j) \qquad (3.13) \\ &= \phi(z_j) - h\ln|u(z_j)| \\ &\quad + \int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z_j,w)\mu(dw) - \int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z_j,w)hn_u(dw) \\ &\le \epsilon_j + \int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z_j,w)\mu(dw). \end{aligned}$$

Harnack's inequality implies that

$$H(z) \le \mathcal{O}(1)(\epsilon_j + \int -G_{\gamma_{\widetilde{r}}}(z_j,w)\mu(dw)) \text{ on } M_j \cap \gamma_{\widehat{r}},\ \widehat{r} = (1-\frac{1}{\widetilde{C}})\widetilde{r}. \qquad (3.14)$$

Now assume that u extends to a holomorphic function in a neighborhood of $\Gamma \cup \overline{\gamma_r}$. We then would like to evaluate the number of zeros of u in Γ. Using (3.8), we first have

$$\#(u^{-1}(0)\cap\gamma_{\widehat{r}}) \le \frac{C}{h}\sum_{j=1}^{N}\left(\epsilon_j + \int_{\gamma_r} -G_{\gamma_r}(z_j,w)\mu(dw)\right). \qquad (3.15)$$

Let $\chi \in C_0^\infty(\Gamma\cup\gamma_{\widehat{r}};[0,1])$ be equal to 1 on Γ. Of course χ will have to depend on r but we may assume that for all $k \in \mathbf{N}$,

$$\nabla^k\chi = \mathcal{O}(r^{-k}). \qquad (3.16)$$

We are interested in

$$\int \chi(z)hn_u(dz) = \int_{\gamma_{\widetilde{r}}} h\ln|u(z)|\Delta\chi(z)L(dz). \qquad (3.17)$$

Here we have on $\gamma_{\widetilde{r}}$

$$\begin{aligned} h\ln|u(z)| &= \Psi(z) - \int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z,w)hn_u(dw) \qquad (3.18) \\ &= \Phi(z) - H(z) - \int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z,w)hn_u(dw) \\ &= \phi(z) + \int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z,w)\mu(dw) - H(z) - \int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z,w)hn_u(dw) \\ &= \phi(z) + R(z), \end{aligned}$$

where the last equality defines $R(z)$.

Inserting this in (3.17), we get

$$\int \chi(z)hn_u(dz) = \int \chi(z)\mu(dz) + \int R(z)\Delta\chi(z)L(dz). \qquad (3.19)$$

(Here we also used some extension of ϕ to Γ with $\mu = \Delta\phi$.) The task is now to estimate $R(z)$ and the corresponding integral in (3.19). Put

$$\mu_j = \mu(M_j \cap \gamma_{\widetilde{r}}). \tag{3.20}$$

Using the exponential decay property (2.26) (equally valid for $G_{\gamma_{\widetilde{r}}}$) we get for $z \in M_j \cap \gamma_{\widetilde{r}}$, $\mathrm{dist}\,(z, \partial M_j) \geq r(z_j)/\mathcal{O}(1)$:

$$\int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z,w)\mu(dw) \leq \int_{M_j\cap\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z,w)\mu(dw) + \mathcal{O}(1)\sum_{k\neq j}\mu_k e^{-\frac{1}{C_0}|j-k|}, \tag{3.21}$$

where $|j-k|$ denotes the natural distance from j to k in $\mathbf{Z}/N\mathbf{Z}$ and $C_0 > 0$. Similarly from (3.14), we get

$$H(z) \leq \mathcal{O}(1)(\epsilon_j + \int_{M_j\cap\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z_j,w)\mu(dw) + \sum_{k\neq j} e^{-\frac{1}{C_0}|j-k|}\mu_k), \tag{3.22}$$

for $z \in M_j \cap \gamma_{\widetilde{r}}$.

This gives the following estimate on the contribution from the first two terms in $R(z)$ to the last integral in (3.19):

$$\begin{aligned} &\int_{\gamma_{\widetilde{r}}} \left(\int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z,w)\mu(dw) - H(z) \right) \Delta\chi(z) L(dz) \qquad (3.23)\\ &= \mathcal{O}(1)\sum_j (\epsilon_j + \int_{M_j\cap\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z_j,w)\mu(dw)) + \sum_{k\neq j} e^{-\frac{1}{C_0}|j-k|}\mu_k)\\ &+\mathcal{O}(1)\sum_j \int_{M_j\cap\gamma_{\widetilde{r}}}\int_{M_j\cap\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z,w)\mu(dw)|\Delta\chi(z)|L(dz). \end{aligned}$$

Here,

$$\int_{M_j\cap\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z,w)|\Delta\chi(z)|L(dz) = \mathcal{O}(1), \tag{3.24}$$

so (3.23) leads to

$$\begin{aligned} &\int_{\gamma_{\widetilde{r}}} \left(\int_{\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z,w)\mu(dw) - H(z) \right) \Delta\chi(z) L(dz) \qquad (3.25)\\ &= \mathcal{O}(1)\left(\mu(\gamma_{\widetilde{r}}) + \sum_j \epsilon_j + \sum_j \int_{M_j\cap\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z_j,w)\mu(dw) \right). \end{aligned}$$

The contribution from the last term in $R(z)$ (in (3.18)) to the last integral in (3.19) is

$$\int_{z\in\gamma_{\widetilde{r}}}\int_{w\in\gamma_{\widetilde{r}}} G_{\gamma_{\widetilde{r}}}(z,w) h n_u(dw)\Delta\chi(z)L(dz). \tag{3.26}$$

Here, by using an estimate similar to (3.21), with $\mu(dw)$ replaced by $L(dz)$, together with (3.24), we get

$$\int_{z\in\gamma_{\widetilde{r}}} G_{\gamma_{\widetilde{r}}}(z,w)(\Delta\chi)(z)L(dz) = \mathcal{O}(1),$$

so the expression (3.26) is by (3.15)

$$\begin{aligned} & \mathcal{O}(h)\#(u^{-1}(0)\cap\gamma_{\widetilde{r}}) \qquad (3.27)\\ = \; & \mathcal{O}(1)\sum_{j=1}^{N}(\epsilon_j + \int_{\gamma_r}(-G_{\gamma_r}(z_j,w))\mu(dw))\\ = \; & \mathcal{O}(1)(\mu(\gamma_r) + \sum_{j=1}^{N}(\epsilon_j + \int_{M_j} -G_{\gamma_r}(z_j,w)\mu(dw))). \end{aligned}$$

This is quite similar to (3.25). Using Proposition 2.2, we have

$$\begin{aligned} & \int_{M_j\cap\gamma_{\widetilde{r}}} -G_{\gamma_{\widetilde{r}}}(z_j,w)\mu(dw) \le \\ & \mathcal{O}(1)(\int_{|w-z_j|\le\frac{r(z_j)}{C}} |\ln\frac{|z_j-w|}{r(z_j)}|\mu(dw) + \mu(M_j\cap\gamma_{\widetilde{r}})) \end{aligned}$$

and similarly for the last integral in (3.27). Using all this in (3.19), we get

$$\begin{aligned} & \int\chi(z)hn_u(dz) = \int\chi(z)\mu(dz) \qquad (3.28)\\ & +\mathcal{O}(1)(\mu(\gamma_r) + \sum_j(\epsilon_j + \int_{|w-z_j|\le r(z_j)/C} |\ln(\frac{|z_j-w|}{r(z_j)})|\mu(dw)). \end{aligned}$$

We replace the smoothness assumption on ϕ by the assumption that ϕ is continuous near Γ and keep (3.6). Then by regularization, we still get (3.28).

Now we observe that

$$|\#(u^{-1}(0)\cap\Gamma) - \frac{1}{2\pi h}\int\chi(z)hn_u(dz)| \le \#(u^{-1}(0)\cap\gamma_{\widetilde{r}}),$$

which can be estimated by means of (3.27), and combining this with (3.28), we get

$$\begin{aligned} & |\#(u^{-1}(0)\cap\Gamma) - \frac{1}{2\pi h}\mu(\Gamma)| \le \qquad (3.29)\\ & \frac{\mathcal{O}(1)}{h}\left(\mu(\gamma_r) + \sum_j(\epsilon_j + \int_{|w-z_j|\le\frac{r(z_j)}{C}} |\ln\frac{|z_j-w|}{r(z_j)}|\mu(dw))\right). \end{aligned}$$

This completes the proof of Theorem 1.1. $\square$

We next discuss when the contribution from the logarithmic integrals in (1.8) can be eliminated or simplified. Let r, C_1, z_j^0 be as in Theorem 1.1. Using the estimates above, we get

$$\int_{D(z_j^0,\frac{r(z_j^0)}{2C_1})}\int_{D(z,\frac{r(z)}{4C_1})}|\ln\frac{|w-z|}{r(z)}|\mu(dw)\frac{L(dz)}{L(D(z_j^0,\frac{r(z_j^0)}{2C_1}))}\le$$
$$\int_{D(z_j^0,\frac{r(z_j^0)}{2C_1})}\int_{D(z_j^0,\frac{r(z_j^0)}{C_1})}|\ln\frac{|w-z|}{r(z)}|\mu(dw)\frac{L(dz)}{L(D(z_j^0,\frac{r(z_j^0)}{2C_1}))}\le$$
$$\mathcal{O}(1)\mu(D(z_j^0,\frac{r(z_j^0)}{C_1})),$$

where we changed the order of integrations in the last step and also used that $r(z)\asymp r(z_j^0)$ in $D(z_j^0,\frac{r(z_j^0)}{C_1})$. We conclude that the mean-value of

$$D(z_j^0,\frac{r(z_j^0)}{2C_1})\ni z\mapsto\int_{D(z,\frac{r(z)}{4C_1})}|\ln\frac{|w-z|}{r(z)}|\mu(dw)$$

is $\mathcal{O}(1)\mu(D(z_j^0,\frac{r(z_j^0)}{C_1}))$. Thus we can find $\widetilde{z}_j\in D(z_j^0,\frac{r(z_j^0)}{2C_1})$ such that

$$\sum_{j=1}^{N}\int_{D(\widetilde{z}_j,\frac{r(\widetilde{z}_j)}{4C_1})}|\ln\frac{|w-\widetilde{z}_j|}{r(\widetilde{z}_j)}|\mu(dw)=\mathcal{O}(1)\mu(\widetilde{\gamma}_r).$$

This gives Theorem 1.2. $\square$

For completeness, we recall (a slight extension of) the counting proposition in [11]. Assume for simplicity that Γ is an h independent Lipschitz domain as defined in the introduction and that ϕ is subharmonic function, defined in a fixed neighborhood of the boundary Assume that $\mu=\Delta\phi$. Let $\rho_0\in]0,2]$ and assume that for all discs $D(z,t)$ contained in a fixed neighborhood of $\gamma=\partial\Gamma$,

$$W_z(t):=\mu(D(z,t))=\mathcal{O}(t^{\rho_0}),\tag{3.30}$$

uniformly with respect to z,t.

Remark 3.2 It is easy to see ([11]) that this assumption on $\Delta\phi$ implies that ϕ is continuous near Γ.

As in [11], we have

Lemma 3.3 *Assume (3.30) for some $\rho_0 \in]0,2]$. Then for $0 < 2t < \tilde{r} \ll 1$ and $z \in \mathbf{C}$ for which $D(z,\tilde{r})$ belongs to the fixed neighborhood of γ, where μ is defined and fulfills (3.30), we have*

$$\int_{D(z,\tilde{r})} |\ln \frac{|z-w|}{\tilde{r}}|\mu(dw) \le \mathcal{O}(1)t^{\rho_0} \ln \frac{\tilde{r}}{t} + \mathcal{O}(1)\ln(\frac{\tilde{r}}{t})\mu(D(z,\tilde{r})). \tag{3.31}$$

Proof. This follows from the estimates,

$$\int_{D(z,\tilde{r})\setminus D(z,t)} |\ln \frac{|z-w|}{\tilde{r}}|\mu(dw) \le \mathcal{O}(1)(\ln \frac{\tilde{r}}{t})\mu(D(z,\tilde{r})),$$

and

$$\begin{aligned}\int_{D(z,t)} |\ln \frac{|z-w|}{\tilde{r}}|\mu(dw) &\le \mathcal{O}(1)\int_0^t \ln \frac{\tilde{r}}{s} dW_z(s) \\ &= \mathcal{O}(1)([\ln(\frac{\tilde{r}}{s})W_z(s)]_0^t + \int_0^t \frac{1}{s}W_z(s)ds) \\ &= \mathcal{O}(1)t^{\rho_0}\ln\frac{\tilde{r}}{t}.\end{aligned}$$

$\square$

Corollary 3.4 *Under the same assumptions, we have for every $N \in \mathbf{N}$:*

$$\int_{D(z,\tilde{r})} |\ln \frac{|z-w|}{\tilde{r}}|\mu(dw) \le \mathcal{O}_N(1)(\tilde{r}^N + \ln(\frac{1}{\tilde{r}})\mu(D(z,\tilde{r}))). \tag{3.32}$$

Proof. We just choose $t = \tilde{r}^M$, $0 < M \in \mathbf{N}$ and use that $\ln \tilde{r}^{-M} = M \ln \tilde{r}^{-1}$. $\square$

We now get:

Theorem 3.5 *Assume that (3.30) holds for all discs $D(z,t)$ contained in some fixed neighborhood of γ. Then under the assumtions of Theorem 1.1, we have for every $N \in \mathbf{N}$:*

$$\begin{gathered}|\#(u^{-1}(0)\cap\Gamma) - \frac{1}{2\pi h}\mu(\Gamma)| \le \qquad (3.33)\\ \frac{\widetilde{C}}{h}\left(\sum_1^N (\epsilon_j + \mathcal{O}_N(r(z_j)^N) + \mathcal{O}_N(1)\int_{\widetilde{\gamma}_r} \ln\frac{1}{r(z)}\mu(dz))\right).\end{gathered}$$

4 Application to sums of exponential functions

Consider the function

$$u(z;h) = \sum_1^N e^{\phi_j(z)/h}, \tag{4.1}$$

where N is finite and ϕ_j are holomorphic in the open set $\Omega \subset \mathbf{C}$ and independent of h for simplicity. Put

$$\psi_j(z) = \Re \phi_j(z), \tag{4.2}$$

let $\Gamma \Subset \Omega$ have C^∞ boundary γ and assume

$$\begin{gathered}\forall x \in \gamma,\ \Psi(x) := \max_j \psi_j(x) \text{ is attained} \\ \text{for at most 2 different values of } j,\end{gathered} \tag{4.3}$$

$$\begin{gathered}\text{If } x \in \gamma,\ \Psi(x) = \psi_j(x) = \psi_k(x),\ j \neq k, \\ \text{then } \nu(x, \partial_x)(\psi_j(x) - \psi_k(x)) \neq 0,\end{gathered} \tag{4.4}$$

where ν denotes the normalized vector field (say positively oriented) that is tangent to γ. We shall see that Theorem 1.2 allows us to determine the number of zeros of u in Γ up to $\mathcal{O}(1)$. This result can be further strengthened by using direct arguments (see for instance [13]), but the purpose of this section is simply to illustrate the results above. We also notice that the results will be valid if u is holomorphic in Ω but with the representation (4.1) and the ϕ_j defined only in a neighborhood of γ.

We shall establish the following result (without any claim of novelty, see [13] as well as [5, 1]). For a closely related old result on entire functions, see [14], Chapter VI, Section 3, Theorem 9, attributed to A. Pfluger [15].

Proposition 4.1 *We have*

$$|\#(u^{-1}(0) \cap \Gamma) - \frac{1}{2\pi h} \int_\Gamma \Delta\Psi(z) L(dz)| = \mathcal{O}(1). \tag{4.5}$$

Here, in the case when ψ_j and Ψ are defined only in a neighborhood of γ, we take any distribution extension of Ψ to a neighborhood of Γ. Notice that near γ the function Ψ is subharmonic and $\Delta\Psi$ is supported by the union of the curves $\gamma_{j,k}$. On each such curve, $\Delta\Psi = \frac{\partial}{\partial n}(\psi_j - \psi_k)|dz|$, where n is the unit normal to $\gamma_{j,k}$, oriented so that $\frac{\partial}{\partial n}(\psi_j - \psi_k) > 0$.

We shall prove Proposition 4.1 by means of Theorem 1.2.

Put

$$\Phi(z)=h\ln(\sum_1^N e^{\psi_j(z)/h}),\ z\in \mathrm{neigh}\,(\gamma), \tag{4.6}$$

so that $\Phi(z)=hf(\frac{\psi_1}{h},..,\frac{\psi_N}{h})$, where

$$f(x)=\ln(\sum_1^N e^{x_j}). \tag{4.7}$$

If we define $\theta_j=e^{x_j}/\sum e^{x_k}$, then $\theta_j>0$, $\theta_1+..+\theta_N=1$, and

$$\partial_{x_j}f(x)=\theta_j, \tag{4.8}$$

$$f''(x)=\mathrm{diag}\,(\theta_j)-(\theta_j\theta_k)_{j,k}. \tag{4.9}$$

For $y\in\mathbf{R}^N$, we have

$$\langle f''(x)y,y\rangle=\sum\theta_jy_j^2-(\sum\theta_jy_j)^2,$$

which is ≥ 0, since the function $t\mapsto t^2$ is convex. Hence f is convex.

We apply this to $\Phi(z)$, now with $\theta_j=e^{\psi_j(z)/h}/\sum_k e^{\psi_k/h}$, and get

$$\partial_z\Phi(z)=\sum\theta_j\partial_z\psi_j,\quad \partial_z=\frac{1}{2}(\partial_{\Re z}+\frac{1}{i}\partial_{\Im z}) \tag{4.10}$$

$$\partial_{\overline{z}}\partial_z\Phi(z)=\frac{1}{h}\langle f''\partial_z\psi,\partial_{\overline{z}}\psi\rangle=\frac{1}{h}(\sum\theta_j|\partial_z\psi_j|^2-|\sum\theta_j\partial_z\psi_j|^2). \tag{4.11}$$

In the last calculation, we also used that ψ_j are harmonic. It follows that Φ is subharmonic near γ. Also notice that

$$\Delta\Phi(z)=\mathcal{O}(h^{-1}), \tag{4.12}$$

and that this estimate can be considerably improved away from the union of the $\gamma_{j,k}$: Assume for instance that $\Psi(z)=\psi_1\ge\max_{j\ne 1}\psi_j+\delta$, where $\delta>0$ and notice that we can take $\delta=C^{-1}d(x)$ with $d(x):=\mathrm{dist}\,(z,\cup\gamma_{j,k})$, by (4.3), (4.4). Then

$$\Phi=h\ln(e^{\psi_1/h}(1+\mathcal{O}(e^{-\delta/h})))=\psi_1+\mathcal{O}(he^{-\delta/h}). \tag{4.13}$$

Further,

$$\theta_1=1+\mathcal{O}(e^{-\delta/h}),\ \theta_j=\mathcal{O}(e^{-\delta/h})\text{ for }j\ne 1, \tag{4.14}$$

so

$$\partial_z\Phi=\partial_z\psi_1+\mathcal{O}(e^{-\delta/h}), \tag{4.15}$$

$$f''(e^{\psi_1/h},..,e^{\psi_N/h}) = \mathcal{O}(e^{-\delta/h}),$$
$$\partial_{\overline{z}}\partial_z\Phi = \mathcal{O}(\frac{1}{h}e^{-\delta/h}). \tag{4.16}$$

We will always be able to express the final result in terms of the simpler function Ψ:

Lemma 4.2 *We have*

$$\int_\Gamma \Delta\Phi L(dz) - \int_\Gamma \Delta\Psi L(dz) = \mathcal{O}(h). \tag{4.17}$$

Proof. Using Green's formula, the left hand side of (4.17) can be written

$$\int_\gamma (\frac{\partial\Phi}{\partial n} - \frac{\partial\Psi}{\partial n})|dz|,$$

where n is the suitably oriented normal direction. It then suffices to apply (4.15), with ψ_1 replaced by Ψ, in the region where $d(z) \gg h$ and use that the gradients of Φ, Ψ are $\mathcal{O}(1)$. $\square$

We next notice that

$$h\ln|u(z;h)| \le \Phi(z) \tag{4.18}$$

in neighborhood of γ. On the other hand, for z near γ, $d(z) \gg h$, we have

$$h\ln|u(z;h)| \ge \Phi(z) - \mathcal{O}(h)e^{-d(z)/(Ch)}. \tag{4.19}$$

We can now apply Theorem 1.2 with $r = \mathrm{Const.}\, h$, $d(z_j^0) \ge Ch$, $\epsilon_j = \mathcal{O}(he^{-d(z_j^0)/(Ch)})$, $\phi = \Phi$. In view of (4.12), (4.16), we see that $\mu(\widetilde{\gamma}_r) = \mathcal{O}(h)$, $\sum\epsilon_j = \mathcal{O}(h)$, so

$$\#(u^{-1}(0)\cap\Gamma) - \frac{1}{2\pi h}\int_\Gamma \Delta\Phi L(dz) = \mathcal{O}(1),$$

and we obtain Proposition 4.1 from Lemma 4.2. $\square$

If we would like to work directly with $\phi = \Psi$, we still have (4.19) with Φ replaced by Ψ, while the upper bound (4.18) has to be replaced by

$$h\ln|u(z)| \le \Psi(z) + Ch,$$

so we have to take $\phi = \Psi + Ch$ and at most places $\epsilon_j \asymp h$. The effect of that deterioration can be limited by chosing the z_j more sparcely away from the union of the $\gamma_{j,k}$, but we can hardly avoid a remainder $\mathcal{O}(\ln\frac{1}{h})$ in (4.5). Similarly, by working with $\phi = \Phi$ and the weaker Theorem 3.5 (essentially from [11]) we also seem to get additional logarithmic losses.

5 A simple application to entire functions

In this section we give as a simple application a result on the number of zeros in truncated sectors for entire functions with a certain exponential growth. We have been inspired by Theorem 3 in Section 3 of Chapter 3 in [14], there attributed to Pfluger [15]. We give a variant which is not quite identical.

Let $\theta < \vartheta < \theta + 2\pi$ and let u be an entire function or more generally a holomorphic function defined in the sector

$$\mathbf{R}_+ e^{i]\theta-\epsilon_0,\vartheta+\epsilon_0[}, \tag{5.1}$$

for some fixed $\epsilon_0 > 0$. Let ϕ be a continuous subharmonic function in the same sector, positively homogeneous of degree $\rho > 0$, so that $\phi = r^\rho g(\omega)$ in polar coordinates $z = re^{i\omega}$. Then

$$\Delta\phi = r^{\rho-2}(\rho^2 g(\omega) + g''(\omega)), \tag{5.2}$$

so the subharmonicity means that

$$\nu := g''(\omega) + \rho^2 g(\omega) \geq 0 \text{ on }]\theta - \epsilon_0, \vartheta + \epsilon_0[, \tag{5.3}$$

in the sense of distributions.

Assume that in the sector (5.1), we have

$$\ln|u(z)| \leq \phi(z) + o(|z|^\rho),\ |z| \to \infty. \tag{5.4}$$

Also assume that for all $0 < \widetilde{\epsilon}, \epsilon \ll 1$ there exists $R(\widetilde{\epsilon}, \epsilon) > 0$, such that for every z in the sector (5.1) with $|z| \geq R(\epsilon, \widetilde{\epsilon})$, there exists $\widetilde{z} = \widetilde{z}(\widetilde{\epsilon}, \epsilon)$ in the same sector such that

$$|\widetilde{z} - z| \leq \epsilon|z|,\ \ln|u(\widetilde{z})| \geq \phi(\widetilde{z}) - \widetilde{\epsilon}|\widetilde{z}|^\rho. \tag{5.5}$$

In the proof below we shall take $\widetilde{\epsilon} = \epsilon^2$ where the exponent 2 could be replaced by any exponent > 1. Write $R(\epsilon)$ instead of $R(\epsilon, \epsilon^2)$.

Proposition 5.1 *We make the assumptions above. Also assume that the positive measure ν in (5.3) does not charge θ, ϑ in the sense that*

$$\nu([\theta - \epsilon, \theta + \epsilon]), \nu([\vartheta - \epsilon, \vartheta + \epsilon]) \to 0,\ \epsilon \searrow 0.$$

Then the number $n(R;\theta,\vartheta)$ of zeros of u in $]1, R[e^{i[\theta,\vartheta]} =: \Gamma(R)$ satisfies

$$n(R;\theta,\vartheta) = \frac{1}{2\pi}(\int_{\Gamma(R)} \Delta(\phi)L(dz) + o(R^\rho)),\ R \to \infty. \tag{5.6}$$

Proof. We apply Theorem 1.1 with $h = 1$, $r(z) = \epsilon|z|$, and $\Gamma = \widetilde{\Gamma}(R)$ equal to the truncated sector $[R(\frac{\epsilon}{2C_1}), R]e^{i[\theta,\vartheta]}$ for $R > R(\frac{\epsilon}{2C_1})$. (Here we notice that our domains and r satisfy the assumptions of the theorem uniformly and C_1 is the corresponding uniform constant there.) We choose z_j^0 as in Theorem 1.1 and let $z_j = \widetilde{z}(s_j^0)$ be as in (5.5) with ϵ replaced by $\epsilon/(2C_1)$, so that

$$|z_j - \widetilde{z}_j| \leq \frac{r(z_j)}{2C_1}, \ \ln|u(z_j)| \geq \phi(z_j) - \frac{\epsilon^2}{4C_1^2}|z_j|^\rho. \tag{5.7}$$

Thus we can take ϵ_j in Theorem 1.1 equal to $\frac{\epsilon^2}{4C_1^2}|z_j|^\rho$.

We have $\widetilde{\gamma}_r = \cup_{z\in\partial\widetilde{\Gamma}(R)} D(z, \epsilon|z|)$ and since $\mu = \Delta\phi(z)L(dz)$, some straight forward estimates show that

$$\begin{aligned} \mu(\widetilde{\gamma}_r) \ &\leq \ C_\rho(\nu([\theta - 2\epsilon, \theta + 2\epsilon]) + \\ &\quad\ \nu([\vartheta - 2\epsilon, \vartheta + 2\epsilon]) + \epsilon\nu(\vartheta - 2\epsilon, \theta + 2\epsilon]))R^\rho \\ &= \ o(1)R^\rho, \quad \epsilon \to 0. \end{aligned} \tag{5.8}$$

Here we also used the assumption that ν does not charge θ and ϑ.

Next we estimate $\sum \epsilon_j$. The number of points z_j^0 on the arc $Re^{i[\theta,\vartheta]}$ is $\mathcal{O}(1/\epsilon)$, so the sum of the corresponding ϵ_j is $\mathcal{O}(\epsilon)R^\rho$. The points z_j^0 on $[\widetilde{R}(\epsilon), R]e^{i\theta}$ can be chosen as a geometric progression so that $|z_j^0| = \widetilde{R}(\epsilon)\exp(\epsilon j/C)$ for a suitable constant $C > 0$. Here $j \geq 0$ is bounded from above by the requirement $\epsilon j/C \leq \ln(R/\widetilde{R}(\epsilon))$, so the corresponding sum of ϵ_j is bounded by $\mathcal{O}(1)\epsilon R^\rho$. The two other parts of the boundary can be treated similarly and we conclude that

$$\sum \epsilon_j = \mathcal{O}(1)\epsilon R^\rho. \tag{5.9}$$

We finally have to treat the logarithmic integrals in (1.8). With $r_j = |z_j|$, $\theta_j = \arg z_j$, we have $r(z_j) = \epsilon r_j$ and the integral can be bounded by

$$\int_{|\omega-\theta_j|\leq\frac{\epsilon}{4C_1}} \int_{|r-r_j|\leq\frac{\epsilon r_j}{4C_1}} |\ln\frac{|re^{i\omega} - r_je^{i\theta_j}|}{\epsilon r_j}|r^{\rho-1}dr\nu(d\omega) \leq$$
$$\int_{|\omega-\theta_j|\leq\frac{\epsilon}{4C_1}} \int_{|r-r_j|\leq\frac{\epsilon r_j}{4C_1}} |\ln\frac{|r - r_j|}{\epsilon r_j}|r^{\rho-1}dr\nu(d\omega)$$

Here the presence of the logarithmic factor does not change the order of magnitude of the last integral with respect to r so we conclude that the integral in (1.8) is bounded by

$$\mathcal{O}(1)\mu(\{z \in \mathbf{C}; |z - |z_j|| \leq \frac{\epsilon|z_j|}{4C_1}, \ |\arg z - \arg z_j| \leq \frac{\epsilon}{4C_1}\})$$

and the domains appearing here have at most a fixed finite number of overlaps at each point. Consequently the sum of the logarithmic integrals in (1.8) can be bounded from above by $\mathcal{O}(1)\mu(\widetilde{\gamma}_r)$.

It now follows from Theorem 1.1 that the number of zeros of u in $[R(\frac{\epsilon}{2C_1}), R]e^{i[\theta,\vartheta]}$ is equal to

$$\frac{1}{2\pi}(\int_{[R(\frac{\epsilon}{2C_1}),R]e^{i[\theta,\vartheta]}} \Delta\phi(z)L(dz) + o(R^\rho))$$

when $\epsilon \to 0$, uniformly when $R \geq R(\frac{\epsilon}{2C_1})$. The proposition follows. $\Box$

Using (5.2), (5.3), we can write the right hand side of (5.6) as

$$\frac{1}{2\pi}R^\rho(o(1) + \rho\int_\theta^\vartheta g(\omega)d\omega + \frac{1}{\rho}(g'(\vartheta) - g'(\theta)).$$

In [14] the case of certain variable ρ, so called proximate orders, is also treated. We believe that the discussion above can be extended to cover that case.

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